Scaling of Lyapunov exponents of coupled chaotic systems

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We develop a statistical theory of the coupling sensitivity of chaos. The effect was first described by Daido [Prog. Theor. Phys. **72**, 853 (1984)]; it appears as a logarithmic singularity in the Lyapunov exponent in coupled chaotic systems at very small couplings. Using a continuous-time stochastic model for the coupled systems we derive a scaling relation for the largest Lyapunov exponent. The singularity is shown to depend on the coupling and the systems' mismatch. Generalizations to the cases of asymmetrical coupling and three interacting oscillators are considered, too. The analytical results are confirmed by numerical simulations.

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I. INTRODUCTION

The dynamics of coupled chaotic systems attracted large interest recently. Many interesting phenomena, in particular different kinds of synchronization, can already be observed in the simplest cases of two interacting chaotic attractors [1–3]. While the synchronization occurs for couplings large enough to suppress a chaos-induced tendency to desynchronization, an interesting anomality in the dynamics happens for very small couplings already. This is the effect of coupling sensitivity of chaos, first observed by Daido [4–7] (see also [8,9]): the dependence of the largest Lyapunov exponent on the coupling parameter ε has a singularity $\sim 1/|\ln \varepsilon|$ for small couplings $\varepsilon \to 0$. The largest Lyapunov exponent thus increases when weak coupling is introduced. This counterintuitive effect has been described as a coupling-induced instability [9,10].

The largest Lyapunov exponent measures the growth rate of infinitesimal perturbations to chaotic trajectories and serves as one of the most important characteristics of chaotic motion, in numerics it is a standard tool for proving the existence of chaos. Moreover, many physically relevant properties of chaos, such as the correlation time, entropy, and synchronization threshold, depend on the largest Lyapunov exponent. Therefore, the coupling sensitivity is not only of theoretical interest.

In this paper we study this effect in detail. We apply an analytical approach based on the modeling of the perturbation dynamics in coupled systems with a set of linear stochastic equations (recently such an approach has been applied to coupled map lattices [10], see Sec. II H for details). For this set we get an analytical expression for the largest Lyapunov exponent, valid for arbitrary coupling and systems' parameter mismatch. This allows us to show that the logarithmic singularity disappears if the interacting systems have different exponents. We also obtain analytic expressions for generalized Lyapunov exponents. The theoretical predictions (Sec. II) are illustrated with numerical calculations of coupled maps and interacting high-dimensional continuous-time systems (Sec. III). Apart from the analytical treatment, we present in Sec. II D simple arguments explaining the singularity form with the help of elementary random-walk dynamics.

A theoretical investigation based on modeling the fluctuations of Lyapunov exponents by random noise has already been undertaken by Daido [7]. In contrast to our approach, it started from discrete-time equations and was limited to the case of coupled identical one-dimensional maps.

II. ANALYTICAL APPROACH

A. Stochastic continuous-time model

In this section we formulate and investigate a stochastic continuous-time model for coupled chaotic systems. First, we neglect the high dimensionality of the interacting chaotic systems and describe linear perturbations in each system with a scalar variable. In other words, we are following the perturbation corresponding to the largest Lyapunov exponent only. Second, we model the fluctuations of the growth rate with a stochastic multiplicative term in the equations of motion. This approach has been succesfully applied in studies of different statistical properties of chaos [3,11]. Summarizing, we propose the two-dimensional system of Langevin equations

$$\frac{du_1}{dt} = [\chi_1(t) + \Lambda_1]u_1 + \varepsilon(u_2 - u_1), \tag{1}$$

$$\frac{du_2}{dt} = [\chi_2(t) + \Lambda_2]u_2 + \varepsilon(u_1 - u_2) \tag{2}$$

as a continuous-time model for the linearized equations of coupled chaotic systems. The following three groups of parameters describe three important ingredients of the dynamics.

(i) *Lyapunov exponents* of uncoupled systems are described by the constants $\Lambda_{1,2}$.

(ii) *Fluctuations of local growth rates* are modeled with the terms $\chi_{1,2}(t)$ which are random processes with zero mean values. In order to be able to apply the powerful theory of the Fokker-Planck equation [12], we assume, furthermore, these processes to be independent, Gaussian, and δ correlated

$$\langle \chi_i \rangle = 0, \quad \langle \chi_i(t)\chi_j(t') \rangle = 2\sigma_i^2 \delta_{ij}\delta(t-t').$$

The parameters $\sigma_{1,2}^2$ describe the fluctuations of local expansion rates in the chaotic systems. The quantities $\sigma_{1,2}^2$ can be set in direct correspondence to the uncoupled chaotic systems, if one calculates the distribution of local (finite-time) Lyapunov exponents [13]. Such a distribution has the asymptotic (for large time intervals T) form

$$\mathrm{Prob}(\lambda_T)\sim e^{-T\phi(\lambda_T)},$$

with a scaling function ϕ having its minimum at the true Lyapunov exponent Λ. For the stochastic model (1) and (2) the local Lyapunov exponents are finite-time averages of the Gaussian δ correlated process, so that their distribution is also Gaussian,

$$\mathrm{Prob}(\lambda_T)\sim e^{-T(\lambda_T-\Lambda)^2(2\sigma)^{-2}}.$$

This means that we in fact use the parabolic approximation of the function ϕ and get the parameter σ^2 from this function:

$$\sigma^{-2}=2\phi''(\Lambda).$$

(iii) *Coupling* is described by the last terms on the right-hand-side; it is proportional to the coupling constant ε. For a while a symmetrical coupling is assumed, the case of asymmetrical coupling is considered in Sec. II F below.

Note that in this formulation we assume the statistical properties of the underlying chaotic motion to be independent of the coupling: the parameters $\Lambda_{1,2}$ and the statistical properties of the fluctuations $\chi_{1,2}$ are ε independent. This assumption is supported by the theory [14], where the invariant measure for weakly coupled systems is constructed using perturbation methods, so that the measure has no singularities in dependence on ε. Thus the theory below is valid as soon as we can neglect ε dependence of the statistical properties of chaos compared to singular ε dependence of the largest Lyapounov exponent.

B. The Fokker-Planck equation and the maximal Lyapunov exponent

Before writing the Fokker-Planck equation for the stochastic system (1) and (2), we perform a transformation to new variables. First we note that for large times and positive coupling ε both variables $u_{1,2}$ have the same sign. Indeed, it is easy to see that the regions $u_1,u_2>0$ and $u_1,u_2<0$ are absorbing ones because for $u_1=0$ we have $\dot{u}_1=\varepsilon u_2$ and for $u_2=0$ we have $\dot{u}_2=\varepsilon u_1$. Thus eventually one observes the state with $u_1u_2>0$ independently of initial conditions. So the transformation

$$v_1=\ln(u_1/u_2),\qquad v_2=\ln(u_1u_2)$$

can be performed, leading to the equations

$$\frac{dv_1}{dt}=\xi_1-2\varepsilon\sinh(v_1)+\Lambda_1-\Lambda_2, \tag{3}$$

$$\frac{dv_2}{dt}=\xi_2+2\varepsilon\cosh(v_1)+\Lambda_1+\Lambda_2-2\varepsilon, \tag{4}$$

where $\xi_1=\chi_1-\chi_2$ and $\xi_2=\chi_1+\chi_2$. One can see that the dynamics of v_1 is v_2 independent, thus, although the noisy forcing terms $\xi_{1,2}$ are no more statistically independent, we can write the Fokker-Planck equation for the probability density $\rho(v_1,t)$ [12]:

$$\dot{\rho}=\left[2\varepsilon\cosh(v_1)+2\varepsilon\sinh(v_1)\frac{\partial}{\partial v_1}-(\Lambda_1-\Lambda_2)\frac{\partial}{\partial v_1}\right.$$
$$\left.+2\sigma^2\frac{\partial^2}{\partial v_1^2}\right]\rho, \tag{5}$$

where $\sigma^2=(\sigma_1^2+\sigma_2^2)/2$.

The stationary solution of Eq. (5) is given by

$$\rho_{\mathrm{stat}}(v_1)=C\exp(lv_1-\varepsilon\sigma^{-2}\cosh v_1), \tag{6}$$

where $l=(\Lambda_1-\Lambda_2/2\sigma^2)$, with the normalization constant C.

Based on the solution (6) we now calculate the largest Lyapunov exponent $\lambda_{\max}$ (below we omit the index, denoting the largest exponent for simplicity as λ), defined by

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\frac{1}{2}\langle\ln(u_1^2+u_2^2)\rangle.$$

The norm $u_1^2+u_2^2$ can be expressed in terms of v_1 and v_2 as

$$\ln(u_1^2+u_2^2)=v_2+\ln(2\cosh v_1).$$

Since one is interested in the large-time limit, the stationary distribution of v_1 may be used. Because $\langle\ln(2\cosh v_1)\rangle_{\rho_{\mathrm{stat}}(v_1)}$ is finite and time independent, the only contribution to the largest Lyapunov exponent comes from v_2. Thus Eq. (4) gives the equation for λ,

$$\lambda=\frac{1}{2}\langle\dot{v}_2\rangle=\varepsilon\langle\cosh v_1\rangle+\frac{1}{2}(\Lambda_1+\Lambda_2-2\varepsilon). \tag{7}$$

The averaging with the stationary distribution $\rho_{\mathrm{stat}}(v_1)$ yields

$$\langle\cosh v_1\rangle=\frac{K_{1-|l|}(\varepsilon/\sigma^2)+K_{1+|l|}(\varepsilon/\sigma^2)}{2K_{|l|}(\varepsilon/\sigma^2)},$$

where K_l are modified Bessel functions (Macdonald functions) [15]. Substituting this in Eq. (7) we obtain a final analytical formula for the largest Lyapunov exponent. We write it in a scaling form,

$$\frac{\lambda-\frac{1}{2}(\Lambda_1+\Lambda_2-2\varepsilon)}{\sigma^2}=\frac{\varepsilon}{\sigma^2}\frac{K_{1-|l|}(\varepsilon/\sigma^2)+K_{1+|l|}(\varepsilon/\sigma^2)}{2K_{|l|}(\varepsilon/\sigma^2)}. \tag{8}$$

This form demonstrates that the essential parameters of the problem are the coupling parameter and the Lyapunov exponents' mismatch normalized to the fluctuation of the exponents: ε/σ^2 and $l=(\Lambda_1-\Lambda_2)/(2\sigma^2)$, correspondingly.

Simplified expressions can be obtained in the following limiting cases.

(*a*) *Small coupling, equal Lyapunov exponents.* According to Eq. (6), if the Lyapunov exponents of two interacting systems are equal, $\Lambda_1=\Lambda_2=\Lambda$, then the parameter l vanishes and we get (cf. [10])

$$\lambda=\varepsilon\frac{K_1(\varepsilon/\sigma^2)}{K_0(\varepsilon/\sigma^2)}+\Lambda-\varepsilon.$$

For small coupling ε/σ^2 the leading term in ε is singular, as it follows from the expansions of K_1 and K_0 [15]:

$$\lambda-\Lambda\approx\frac{\sigma^2}{|\ln(\varepsilon/\sigma^2)|}. \tag{9}$$

This formula corresponds to Daido's singular dependence of the Lyapunov exponent on the coupling parameter [4–6]. It is valid in all cases, when identical chaotic systems are coupled symmetrically, provided that the Lyapunov exponents in these systems fluctuate ($\sigma^2>0$). Moreover, even for different systems having however equal Lyapunov exponents (but not necessarily equal fluctuations of the exponents) we get the same singularity as for identical systems. Daido arrived at a similar result in his analytical treatment of coupled one-dimensional maps, cf. Eq. (19) of Ref. [7].

(*b*) *Small coupling, different Lyapunov exponents.* The expansion (9) remains valid for small values of mismatch $|l|$, if $(\varepsilon/\sigma^2)^{|l|}$ is close to 1. For larger mismatch, when

$$\left(\frac{\varepsilon}{\sigma^2}\right)^{|l|}\ll 1,$$

the largest Lyapunov exponent is

$$\lambda\approx 2\sigma^2|l|\frac{\Gamma(1-|l|)}{\Gamma(1+|l|)}\left(\frac{\varepsilon}{2\sigma^2}\right)^{2|l|}+\frac{1}{2}(|\Lambda_1-\Lambda_2|+\Lambda_1+\Lambda_2). \tag{10}$$

The singularity is now of the power-law type, with the power depending on the systems' mismatch. Note also that this is the correction to the largest of the Lyapunov exponents of uncoupled systems $\Lambda_{1,2}$.

(*c*) *Large coupling.* For $\varepsilon/\sigma^2\gg 1$ the expansion of Eq. (8) gives

$$\lambda\approx\frac{\sigma^2}{2}-\frac{(1+3l^2)\sigma^4}{8\varepsilon}+\frac{1}{2}(\Lambda_1+\Lambda_2). \tag{11}$$

C. Generalized Lyapunov exponents

The generalized Lyapunov exponents characterize finite-time fluctuations of the exponential growth rate. For our linear model (1) and (2) they are defined as [13]

$$L(q)=\lim_{t\to\infty}\frac{1}{t}\ln\langle(u_1^2+u_2^2)^{q/2}\rangle. \tag{12}$$

For simplicity of presentation we assume below that the interacting systems are identical and therefore will omit the index at the parameters σ^2 and Λ.

It is straightforward to obtain the generalized Lyapunov exponents for integer q. For $q=1$ we need equations for the mean values $\langle u_{1,2}\rangle$ which can easily be obtained by direct averaging of the system (1) and (2) using the Furutsu-Novikov relation [16,17]

$$\frac{d}{dt}\langle u_1\rangle=(\Lambda+\sigma^2-\varepsilon)\langle u_1\rangle+\varepsilon\langle u_2\rangle, \tag{13}$$

$$\frac{d}{dt}\langle u_2\rangle=(\Lambda+\sigma^2-\varepsilon)\langle u_2\rangle+\varepsilon\langle u_1\rangle. \tag{14}$$

Thus the averages $\langle u_{1,2}\rangle$ grow exponentially and the generalized Lyapunov exponent is

$$L(1)=\Lambda+\sigma^2. \tag{15}$$

Similarly, we can write the three-dimensional system of linear equations for the moments $\langle u_1^2\rangle,\langle u_2^2\rangle,\langle u_1u_2\rangle$ and determine $L(2)$ as the largest eigenvalue of this system

$$L(2)=2\Lambda+3\sigma^2-2\varepsilon+\sqrt{\sigma^4+4\varepsilon^2}. \tag{16}$$

This method works for all integer moments, but for $q>2$ we have to look for roots of polynomials of order 4 and higher, so the analytical expressions are hardly available. Also, we do not have a method for calculation of the generalized exponents for noninteger indices.

Having expressions for $L(1)$ and $L(2)$, we can find an approximate expression for the usual Lyapunov exponent (cf. [18]). Indeed, this exponent is determined by the behavior of $L(q)$ near $q=0$,

$$\lambda=L'(0)$$

[this formula follows directly from Eq. (12), see also [13]]. As $L(q)$ is a convex function and $L(0)=0$, knowing two points $L(1)$ and $L(2)$ we can approximate it with a parabola

$$L(q)=\alpha q+\beta q^2,$$

with the parameters

$$\alpha=2L(1)-\frac{L(2)}{2},\quad \beta=-L(1)+\frac{L(2)}{2}.$$

Thus we get the approximation for the usual Lyapunov exponent

$$\tilde{\lambda}=\alpha=\Lambda+\frac{\sigma^2}{2}+\varepsilon-\sqrt{\frac{\sigma^4}{4}+\varepsilon^2}.$$

For $\varepsilon\gg\sigma^2$ this gives

$$\tilde{\lambda}\approx\frac{\sigma^2}{2}-\frac{\sigma^4}{8\varepsilon}+\Lambda,$$

which coincides with Eq. (11). We see that the parabolic approximation for the generalized exponent spectrum $L(q)$ is valid for large couplings and small fluctuations of the

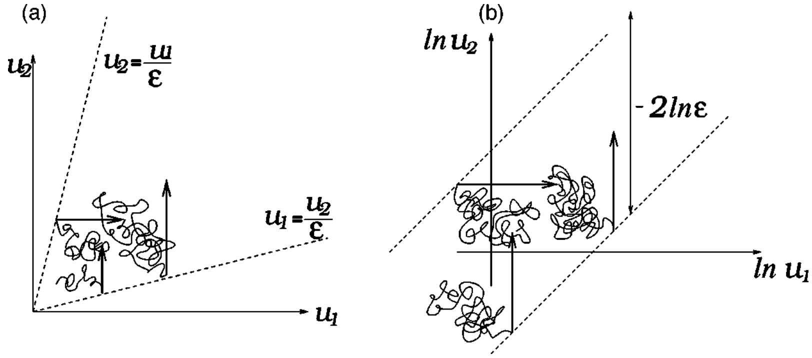

FIG. 1. A sketch of the perturbation dynamics in coupled systems. The curly line shows the random walk not influenced by coupling; straight arrows demonstrate the effect of coupling.

finite-time exponents. Another limiting case, when the form of the generalized exponent spectrum is exactly parabolic, is that of zero coupling

$$L(q)=\Lambda q+\sigma^2 q^2.$$

For small coupling, where the logarithmic singularity of the usual exponent (9) is essential, the parabolic approximation does not work.

D. A qualitative picture

Here we give qualitative arguments supporting the main singularity formula (9). Let us consider the symmetric case and small couplings (moreover, for simplicity of presentation we assume $\Lambda=0$). For small ε, the coupling in the system (1) and (2) influences the dynamics only if the difference between u_1 and u_2 is large. E.g., if $u_2 \sim u_1/\varepsilon \gg u_1$, then the coupling term in the first equation (1) is of the same order as other terms and it contributes to the growth of the variable u_1. At the same time the influence of u_1 on u_2 remains small. Thus, the coupling ''switches on'' only rarely, but leads to effective equalization of the variables where the smallest one is adjusted to the largest one. We illustrate this process in Fig. 1(a).

To make these arguments quantitative, let us represent the same qualitative picture in the plane of logarithmic variables $\ln u_1$, $\ln u_2$ [see Fig. 1(b)]. Here we have a random walk in two dimensions, and this walk is restricted to the strip $|\ln u_1 - \ln u_2| < -\ln\varepsilon$ (the connection between the dynamics in the plane of logarithmic variables and the random walk has already been pointed out in Ref. [7]). The walk has rather strange properties of reflection at the boundaries: it springs to the diagonal $\ln u_1 = \ln u_2$, always in the direction of growth of $\ln u_1$ and $\ln u_2$. Due to these reflections a constant drift arises, whose velocity is easy to estimate. Indeed, for an unbiased random walk starting at the center of the strip the mean time to reach the boundary is $(\ln\varepsilon)^2/\sigma^2$ [19], and this is a characteristic time between reflections. Each reflection makes a contribution of order of $|\ln\varepsilon|$ to the mean drift. So for the mean drift velocity we get $\sigma^2/|\ln\varepsilon|$ in accordance with Eq. (9).

E. The second Lyapunov exponent

From the Fokker-Planck equation approach we have obtained the largest Lyapunov exponent. The second exponent can be found as follows. For the stochastic system (1) and (2) the mean divergence of the phase volume is

$$\left\langle \frac{d}{dt}\ln V\right\rangle = \left\langle \frac{\partial \dot u_1}{\partial u_1}+\frac{\partial \dot u_2}{\partial u_2}\right\rangle = \Lambda_1+\Lambda_2-2\varepsilon,$$

and this quantity is just the sum of the Lyapunov exponents. Thus

$$\lambda_2=-\lambda_1+\Lambda_1+\Lambda_2-2\varepsilon, \tag{17}$$

and we get for λ_2 the same singularity as for the largest exponent, only with another sign.

F. Asymmetrical coupling

The more general case of asymmetrical coupling can be described by the following set of Langevin equations:

$$\frac{du_1}{dt}=[\chi_1(t)+\Lambda_1]u_1+\varepsilon_1(u_2-u_1), \tag{18}$$

$$\frac{du_2}{dt}=[\chi_2(t)+\Lambda_2]u_2+\varepsilon_2(u_1-u_2). \tag{19}$$

By virtue of the scaling transformation

$$\tilde u_1=\sqrt{\varepsilon_2}u_1, \quad \tilde u_2=\sqrt{\varepsilon_1}u_2, \tag{20}$$

the problem can be reduced to the symmetric case,

$$\dot{\tilde u}_1=(\chi_1+\Lambda_1-\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_2})\tilde u_1+\sqrt{\varepsilon_1\varepsilon_2}(\tilde u_2-\tilde u_1),$$

$$\dot{\tilde u}_2=(\chi_2+\Lambda_2-\varepsilon_2+\sqrt{\varepsilon_1\varepsilon_2})\tilde u_2+\sqrt{\varepsilon_1\varepsilon_2}(\tilde u_1-\tilde u_2).$$

Thus, we can use the expression (8) for the largest Lyapunov exponent, leading to

$$\frac{\lambda_1-\frac{1}{2}(\Lambda_1+\Lambda_2-\varepsilon_1-\varepsilon_2)}{\sigma^2} = \frac{\sqrt{\varepsilon_1\varepsilon_2}}{\sigma^2}\,\frac{K_{1-|l|}(\sqrt{\varepsilon_1\varepsilon_2}/\sigma^2)+K_{1+|l|}(\sqrt{\varepsilon_1\varepsilon_2}/\sigma^2)}{2K_{|l|}(\sqrt{\varepsilon_1\varepsilon_2}/\sigma^2)}. \tag{21}$$

Here the effective mismatch and the effective coupling are now given by

$$l=\frac{1}{2\sigma^2}[(\Lambda_1-\varepsilon_1)-(\Lambda_2-\varepsilon_2)], \quad \varepsilon=\sqrt{\varepsilon_1\varepsilon_2}.$$

In the case of unidirectional coupling the ansatz (20) is no more valid, but in this situation the Lyapunov exponents can easily be found directly. If, e.g., $\varepsilon_1=0$, then the Lyapunov exponents are Λ_1, $\Lambda_2-\varepsilon_2$. There is no singularity for unidirectional coupling.

The results for asymmetrical coupling can straightforwardly be understood in the framework of the qualitative picture of Sec. II D. Indeed, the important quantity is the width of the strip in Fig. 1(b), and this is $-(\ln\varepsilon_1+\ln\varepsilon_2)$. In the limiting case of unidirectional coupling the width tends to infinity, the random walk never hits the boundary, and there are no essential corrections to the uncoupled exponents.

G. Three coupled chaotic systems

Models with many coupled identical chaotic systems have attracted large attention recently (e.g., [20,21]). As a first step in this direction, we show here that the system of three symmetrically coupled oscillators has the same logarithmic singularity in the Lyapunov exponent as the system of two oscillators. The stochastic model has the following form:

$$\frac{du_1}{dt}=[\chi_1(t)+\Lambda]u_1+\varepsilon(u_2+u_3-2u_1),$$

$$\frac{du_2}{dt}=[\chi_2(t)+\Lambda]u_2+\varepsilon(u_1+u_3-2u_2), \tag{22}$$

$$\frac{du_3}{dt}=[\chi_3(t)+\Lambda]u_3+\varepsilon(u_1+u_2-2u_3),$$

with uncorrelated noisy terms χ_i. In the variables

$$v_1=\ln\frac{u_1}{u_3},\quad v_2=\ln\frac{u_2}{u_3},\quad v_3=\ln(u_1u_2u_3)$$

the system (22) reduces to

$$\frac{dv_1}{dt}=\chi_1-\chi_3-2\varepsilon\sinh(v_1)-\varepsilon e^{v_2}+\varepsilon e^{v_2-v_1},$$

$$\frac{dv_2}{dt}=\chi_2-\chi_3-2\varepsilon\sinh(v_2)-\varepsilon e^{v_1}+\varepsilon e^{v_1-v_2}, \tag{23}$$

$$\frac{dv_3}{dt}=\chi_1+\chi_2+\chi_3+3(\Lambda-2\varepsilon)+2\varepsilon\cosh(v_1-v_2)$$
$$+2\varepsilon\cosh(v_1)+2\varepsilon\cosh(v_2).$$

As the equations for v_1,v_2 constitute a closed system, we can write the Fokker-Planck equation for the probability density $\rho(v_1,v_2)$. In the limit $\varepsilon\to0$ the stationary solution of this equation can be looked for in the ''quasipotential'' (see, e.g., [22]) form $\rho=C\exp[(\varepsilon/\sigma^2)f(v_1,v_2)]$, which in the first order in ε gives (see [23] for details)

$$\rho=C\exp\left[-\frac{\varepsilon}{\sigma^2}[\cosh(v_1-v_2)+\cosh(v_1)+\cosh(v_2)]\right]. \tag{24}$$

The largest Lyapunov exponent can, analogous to Eq. (7), be represented as

$$\lambda=\Lambda-2\varepsilon+\frac{2\varepsilon}{3}\langle\cosh(v_1-v_2)+\cosh(v_1)+\cosh(v_2)\rangle.$$

The averaging requires a nontrivial integral, which can be estimated in the limit $\varepsilon\sigma^{-2}\to0$ (see [23]) to give the final formula

$$\lambda-\Lambda\sim\frac{4}{3}\frac{\sigma^2}{|\ln(\varepsilon/\sigma^2)|}. \tag{25}$$

The singularity in the system of three coupled chaotic oscillators is thus of the same type as for two oscillators, cf. Eq. (9).

H. Coupled map lattices

In a parallel work, an approach similar to ours has been used to derive an analytic expression for the scaling of the largest Lyapunov exponent in coupled map lattices [10]. For small coupling, coupled identical maps, and positive multipliers, the authors arrive at an expression

$$\lambda-\Lambda\approx\frac{\sigma^2}{|\ln(\gamma/\sigma^2)|},$$

where $\gamma=\varepsilon/(1-2\varepsilon)$. This result is similar to ours for the case of two coupled systems, Eq. (9), with the difference that ε is replaced by γ as the scaling parameter.

The authors of Ref. [10] were also able to derive an expression for the case of multipliers with fluctuating signs,

$$\lambda-\Lambda\approx\frac{3}{2}\frac{\sigma^2}{|\ln(\gamma/\sigma)|}.$$

Instead of the variance σ^2, the standard deviation σ appears in the argument of the logarithm.

III. NUMERICAL RESULTS

We now compare the results obtained for the system of continuous-time Langevin equations with numerical calculations for both continuous- and discrete-time deterministic systems. For the calculation of Lyapunov exponents we iterate the original as well as a set of linearized equations, and reorthonormalize the difference vectors periodically using a modified Gram-Schmidt algorithm (see [24,25] and references therein).

For the first examples (Sec. III A–III C) we iterate a system of two diffusively coupled one-dimensional maps,

$$x_1(n+1)=f_1(x_1(n))+\varepsilon[f_2(x_2(n))-f_1(x_1(n))], \tag{26}$$

$$x_2(n+1)=f_2(x_2(n))+\varepsilon[f_1(x_1(n))-f_2(x_2(n))], \tag{27}$$

where $f_1(x)$ and $f_2(x)$ are maps specified below. The linearized equations read

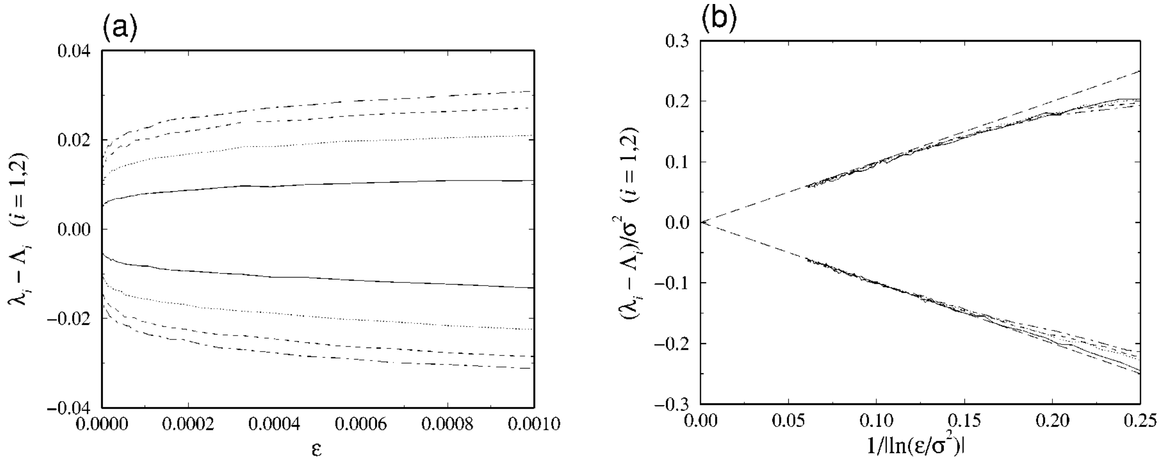

FIG. 2. Coupled skewed Bernoulli maps, Eq. (30). (a) The Lyapunov exponents $\lambda_1-\Lambda_1$ and $\lambda_2-\Lambda_2$ vs ε for $x_0=1/3$ (solid lines), $x_0=1/4$ (dotted lines), $x_0=1/5$ (dashed lines), and $x_0=1/6$ (dash-dotted lines). (b) The same graphs in scaled coordinates. The long-dashed lines show the analytical results $(\lambda_1-\Lambda_1)/\sigma^2=1/|\ln(\varepsilon/\sigma^2)|$ and $(\lambda_2-\Lambda_2)/\sigma^2=-1/|\ln(\varepsilon/\sigma^2)|$, see Eqs. (9) and (17).

$$w_1(n+1)=(1-\varepsilon)f_1'(x_1(n))w_1(n)+\varepsilon f_2'(x_2(n))w_2(n), \tag{28}$$

$$w_2(n+1)=(1-\varepsilon)f_2'(x_2(n))w_2(n)+\varepsilon f_1'(x_1(n))w_1(n). \tag{29}$$

The Lyapunov exponents of the uncoupled maps are $\Lambda_{1,2}=\langle \ln|f_{1,2}'|\rangle$. In the simplest examples considered below the variances can be calculated as $\sigma_{1,2}^2=\langle(\ln|f_{1,2}'|-\Lambda_{1,2})^2\rangle/2$. To have a good correspondence to the theory, we use only monotonous mappings (i.e., with a constant sign of f') below, so that the fluctuations of the local expansion rate are the only source of irregularity of perturbations' dynamics. Another source of irregularity could be irregular changes of the sign of the derivative f' (as for the logistic and the tent maps). Such an irregularity is not covered by our continuous-time approach, but also leads to the logarithmic singularity of type (9), see [10].

A. Skewed Bernoulli maps

We first choose $f_1(x)=f_2(x)=f(x)$ to be identical maps, where $f(x)$ is the skewed Bernoulli map defined as

$$f(x)=\begin{cases} x/x_0 & \text{if} \quad x\leqslant x_0 \\ (x-x_0)/(1-x_0) & \text{if} \quad x>x_0, \end{cases} \tag{30}$$

with $x\in[0,1]$ and $0<x_0<1$. For the uncoupled map, the Lyapunov exponent and the magnitude of fluctuations are given by

$$\Lambda=-x_0\ln x_0-(1-x_0)\ln(1-x_0) \tag{31}$$

and

$$\sigma^2=\frac{1}{2}x_0(1-x_0)\left(\ln\frac{x_0}{1-x_0}\right)^2, \tag{32}$$

respectively. For $x_0=1/2$ we obtain the ordinary Bernoulli map. In this case, there are no fluctuations of the local multipliers ($\sigma^2=0$), and no coupling sensitivity of the Lyapunov exponents is observed.

Figure 2(a) shows the differences $\lambda_{1,2}-\Lambda_{1,2}$ vs ε for maps with different values of $x_0\neq 1/2$. From Fig. 2(b) it can be seen that different curves collapse onto single lines for both exponents when plotted in the rescaled form according to Eq. (8), namely as $(\lambda_1-\Lambda_1)/\sigma^2$ vs $1/|\ln(\varepsilon/\sigma^2)|$. The resulting lines are in very good agreement with the leading term of the theoretical prediction $(\lambda_1-\Lambda_1)/\sigma^2=1/|\ln(\varepsilon/\sigma^2)|$, which is also shown.

No such good accordance between theory and numerical experiment is found in the case of generalized Lyapunov exponents. In Figs. 3(a) and 3(b) the results for $L(1)$ and $L(2)$ are shown for small values of ε. Also shown are the theoretical predictions from Eqs. (15) and (16), respectively. The rough correspondence is completely lost for larger values of ε, although the considerations in Sec. II C are not restricted to small ε.

Much better results are achieved if the derivatives f_i' in the linearized equations (28) and (29) are replaced by independent and identically distributed Gaussian stochastic variables ξ_i $(i=1,2)$. Then the system of equations reads

$$w_1(n+1)=(1-\varepsilon)e^{\xi_1(n)}w_1(n)+\varepsilon e^{\xi_2(n)}w_2(n), \tag{33}$$

$$w_2(n+1)=(1-\varepsilon)e^{\xi_2(n)}w_2(n)+\varepsilon e^{\xi_1(n)}w_1(n), \tag{34}$$

with $\langle\xi_i(n)\rangle=\Lambda$ and $\langle(\xi_i(n)-\Lambda)(\xi_j(m)-\Lambda)\rangle=2\sigma^2\delta_{ij}\delta_{nm}(i,j=1,2)$. In Figs. 4(a) and 4(b) the results for $L(1)$ and $L(2)$, respectively, are shown together with the analytical curves. The values of Λ and σ^2 were calculated by means of Eqs. (31) and (32) with the values of x_0 used above for the skewed Bernoulli map.

An explanation for the discrepancy between the deterministic and stochastic results is that the distribution of $f'(x_i)$ is changed with increasing ε, while the distribution of the stochastic variables ξ_i remains constant. Furthermore, $f'(x_1)$ and $f'(x_2)$ are not statistically independent for larger values of ε. These effects have no observable influence in the case of usual Lyapunov exponents (Fig. 2) because of the singularity. In the case of generalized Lyapunov exponents, however, the nonsingular scaling functions are much more sensitive against changes in the distribution of multipliers.

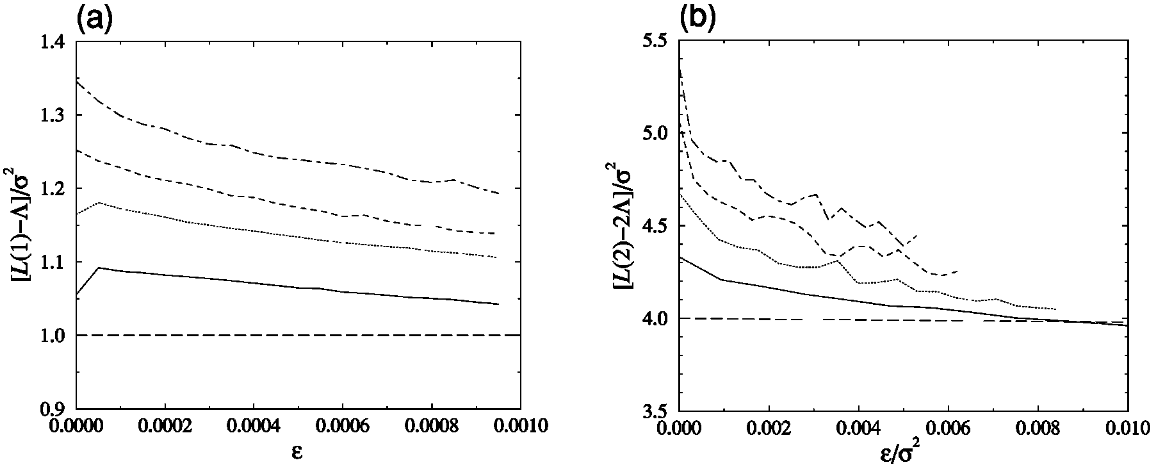

FIG. 3. Generalized Lyapunov exponents for the skewed Bernoulli maps. (a) Rescaled exponent $[L(1)-\Lambda]/\sigma^2$ vs ε for the same values of x_0 as in Fig. 2. The long-dashed line shows the analytical result $[L(1)-\Lambda]/\sigma^2=1$, see Eq. (15). (b) Rescaled exponent $[L(2)-2\Lambda]/\sigma^2$ vs ε/σ^2 for the same values of x_0 as in Fig. 2. The long-dashed line shows the analytical result $[L(2)-2\Lambda]/\sigma^2=3-2\varepsilon/\sigma^2+\sqrt{1+4(\varepsilon/\sigma^2)^2}$, see Eq. (16).

B. Different maps

One main result of the analytical approach is that the singularity does only depend on the average $\sigma^2=(\sigma_1^2+\sigma_2^2)/2$ of the fluctuations of local expansion rates and on the mismatch $l=(\Lambda_1-\Lambda_2)/(2\sigma^2)$ of the Lyapunov exponents of the uncoupled systems. Although no singularity occurs if $\sigma^2=0$, we can expect to observe coupling sensitivity in the case of a system with fluctuations $(\sigma_1^2>0)$ coupled to one without fluctuations $(\sigma_2^2=0)$, given that the mismatch l is sufficiently small.

In order to check this prediction, we again numerically iterate the system of Eqs. (26)–(29), now choosing two different maps. The first map is again the skewed Bernoulli map $[f_1(x)=f(x)$, see Eq. (30)], while the second map is defined as

$$f_2(x)=e^{\Lambda_1}x, \quad (\mathrm{mod}\ 1), \tag{35}$$

where Λ_1 is the Lyapunov exponent of the skewed Bernoulli map $f(x)$ [see Eq. (31)]. With this choice we have the parameters $\sigma_1^2>0$, $\sigma_2^2=0$, and $l=0$.

In Fig. 5 the result is compared with the previous result for two coupled identical skewed Bernoulli maps ($x_0=1/4$ in either case). As expected, the logarithmic singularity is observed in both cases, although the deviation $|\lambda_i-\Lambda_i|$ is smaller if $\sigma_2^2=0$. When rescaled with the average σ^2, however, the curves collapse onto single lines for the first and second Lyapunov exponents, as can be seen in Fig. 5(b).

C. Systems with anomalous fluctuations of Lyapunov exponents

Daido found out that for coupled logistic maps $f(x)=4x(1-x)$ the Lyapunov exponents exhibit power-law instead of logarithmic singular behavior due to anomalous fluctuations of the finite-time Lyapunov exponents [7]. Here we report a similar observation in the case of coupled strange nonchaotic attractors.

Fluctuations of finite-time Lyapunov exponents is a typical feature of chaotic systems, but in some nonchaotic systems the Lyapunov exponents fluctuate as well. To this class belong strange nonchaotic attractors (SNAs) that have a

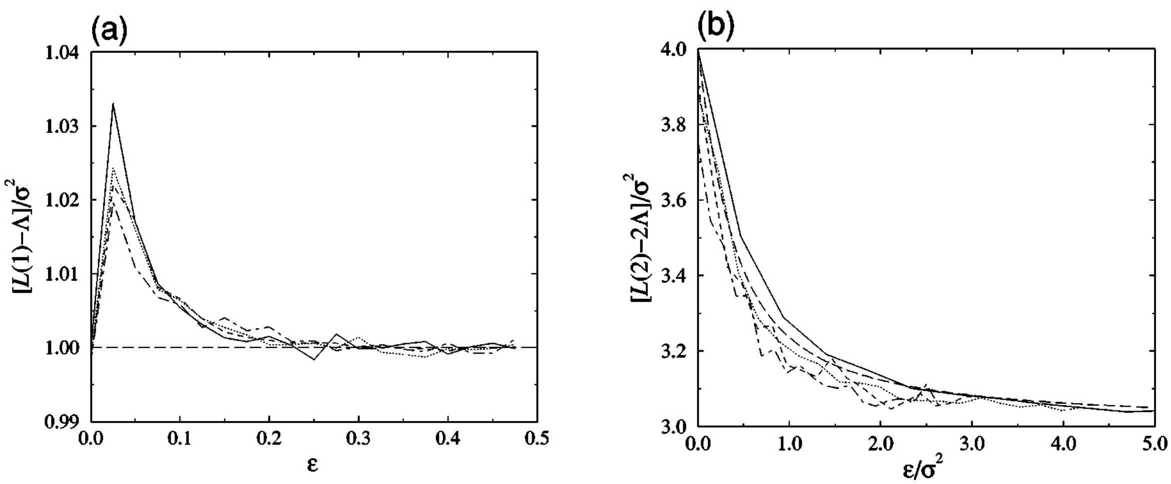

FIG. 4. Rescaled generalized Lyapunov exponents in stochastic maps. (a) The exponent $[L(1)-\Lambda]/\sigma^2$ vs ε for Λ and σ^2 corresponding to the values of x_0 used for Fig. 2. The long-dashed line shows the analytical result as in Fig. 3(a). (b) The exponent $[L(2)-2\Lambda]/\sigma^2$ vs ε/σ^2 for Λ and σ^2 corresponding to the values of x_0 used for Fig. 2. The long-dashed line shows the analytical result as in Fig. 3(b).

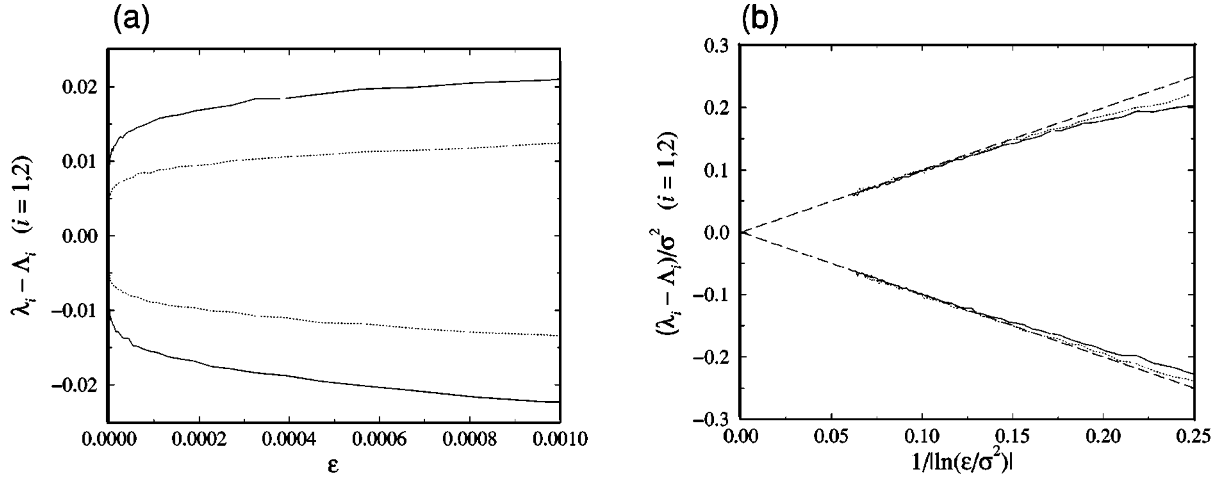

FIG. 5. Different maps. (a) $\lambda_1 - \Lambda_1$ and $\lambda_2 - \Lambda_2$ vs ε for two coupled skewed Bernoulli maps with $x_0 = 1/4$ (solid lines) as well as one skewed Bernoulli map with $x_0 = 1/4$ coupled with the different map (35) (dotted lines). (b) $(\lambda_1 - \Lambda_1)/\sigma^2$ and $(\lambda_2 - \Lambda_2)/\sigma^2$ vs $1/|\ln(\varepsilon/\sigma^2)|$ for the same examples as in Fig. 5(a). The long-dashed lines show the analytical results as in Fig. 2(b).

negative maximal Lyapunov exponent but a complex fractal structure in the phase space (see [26] and references there). The fluctuations of finite-time Lyapunov exponents are present in SNAs [26], but they are much more correlated than in chaotic systems. We demonstrate below that this leads to weaker singularity in the Lyapunov exponent dependence on coupling.

We studied numerically two coupled quasiperiodically forced maps having strange nonchaotic attractors, taking

$$f(x) = 2.5\tanh(x)|\sin(\omega n + \phi)|, \tag{36}$$

where $\omega = (\sqrt{5} - 1)/2$ is the frequency of quasiperiodic driving. The model (36) has been studied rigorously in [27,28]. The results are presented in Fig. 6. The dependence of the Lyapunov exponents on the coupling has a singularity, but this singularity contrary to Eq. (9) is a power law, with a power close to 1/2. A detailed theory needs correct account of nontrivial correlation properties of the SNA and is now in progress.

D. High-dimensional continuous-time systems

Daido observed the effect of coupling sensitivity of chaos not only for coupled one-dimensional maps, but also for two-dimensional discrete-time maps [6]. Here we give numerical evidence that the logarithmic singularity is also observed in infinite-dimensional and continuous-time systems. As an example we study a system of two coupled one-dimensional delay differential equations. A delay differential equation has an infinite number of Lyapunov exponents, and for large delays usually a finite number of exponents is positive. The system we study reads

$$\dot{x}_1(t) = f(x_1(t), x_1(t-\tau)) + \varepsilon[x_2(t) - x_1(t)], \tag{37}$$

$$\dot{x}_2(t) = f(x_2(t), x_2(t-\tau)) + \varepsilon[x_1(t) - x_2(t)], \tag{38}$$

where

$$f(x(t), x(t-\tau)) = -x(t) + a\sin x(t-\tau)$$

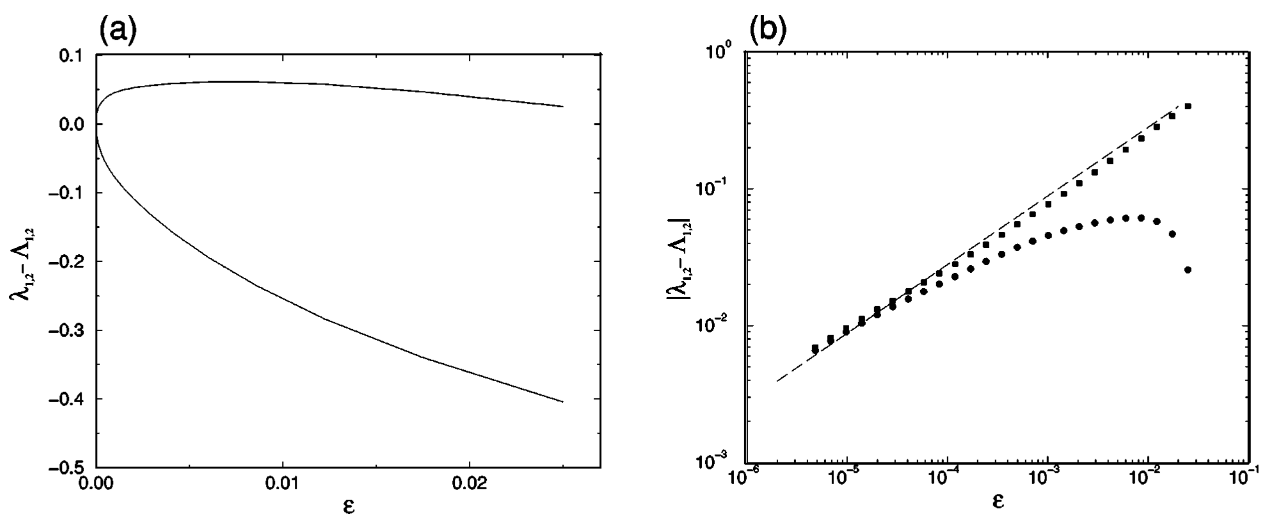

FIG. 6. The Lyapunov exponents in coupled strange nonchaotic attractors in natural coordinates (a) and in a log-log representation (b). The dashed line in (b) has slope 0.5.

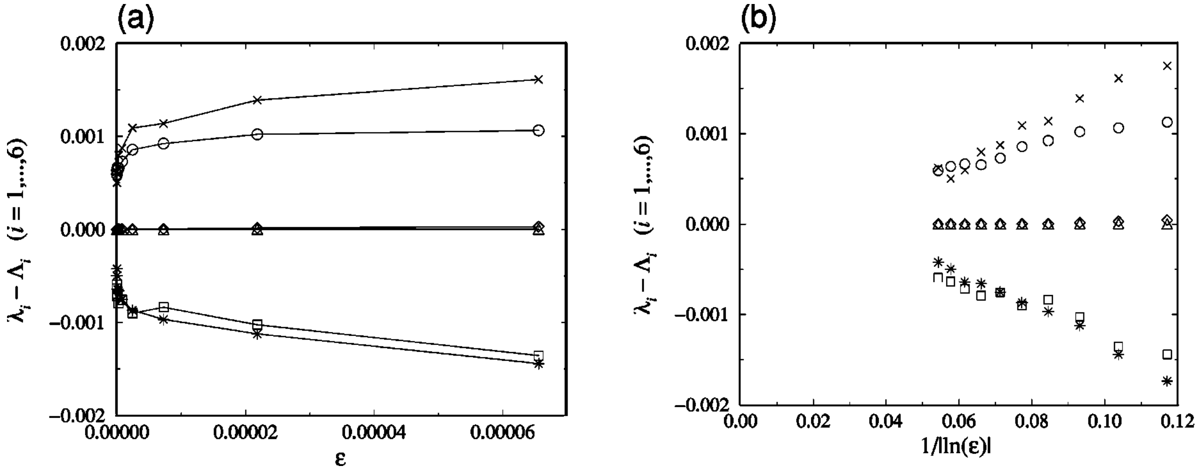

FIG. 7. The Lyapunov exponents in the coupled Ikeda equations, in natural (a) and scaled (b) coordinates. Open circle and open square: the splitting of the positive Lyapunov exponent; open triangle and open rhombohedral: the splitting of the zero exponent; cross and star: the splitting of the closest to zero negative exponent.

corresponds to the Ikeda equation, describing an optical resonator system [29]. The parameter values were chosen to be $a=3.0$ and $\tau=5.0$. We integrated the coupled Ikeda equations, together with the linearized equations, using the fourth-order Runge-Kutta routine. The results are presented in Figs. 7(a) and 7(b). The uncoupled Ikeda system has one positive and one zero (due to invariance to time shifts) Lyapunov exponent, all other exponents are negative. In the coupled system the two former zero exponents (the third and the fourth) are not affected by the coupling sensitivity: one exponent remains exactly zero, changes of the another one are hardly seen for small couplings. We attribute this to the fact that the zero Lyapunov exponent in an autonomous system does not fluctuate. The other Lyapunov exponents (the positive one and the first negative one), however, show the logarithmic singularity.

E. Three coupled chaotic maps

For three weakly coupled chaotic systems, the leading terms in the expressions for the maximum and minimum Lyapunov exponents were shown to have the same logarithmic singularities as in the case of two coupled systems, although with a different factor (see Sec. II G). The singularity is observed in numerical simulations for three coupled identical skewed Bernoulli maps [see Fig. 8(a)]. The factor of 4/3, however, is obviously not correct, although a rough agreement between theoretical and numerical results can be seen in Fig. 8(b). A reason for the disagreement could be the neglect of terms of order ε^2 when finding the stationary probability distribution, Eq. (24).

IV. CONCLUSION

In this paper we used the Langevin approach to obtain statistical properties of the Lyapunov exponents for small coupling. For the simplest system of two coupled stochastic equations it is possible to obtain an analytical expression for the largest Lyapunov exponent, for different values of parameters (coupling, Lyapunov exponents of uncoupled systems, fluctuations of Lyapunov exponents). The logarithmic singularity, first discovered by Daido, is shown to exist even

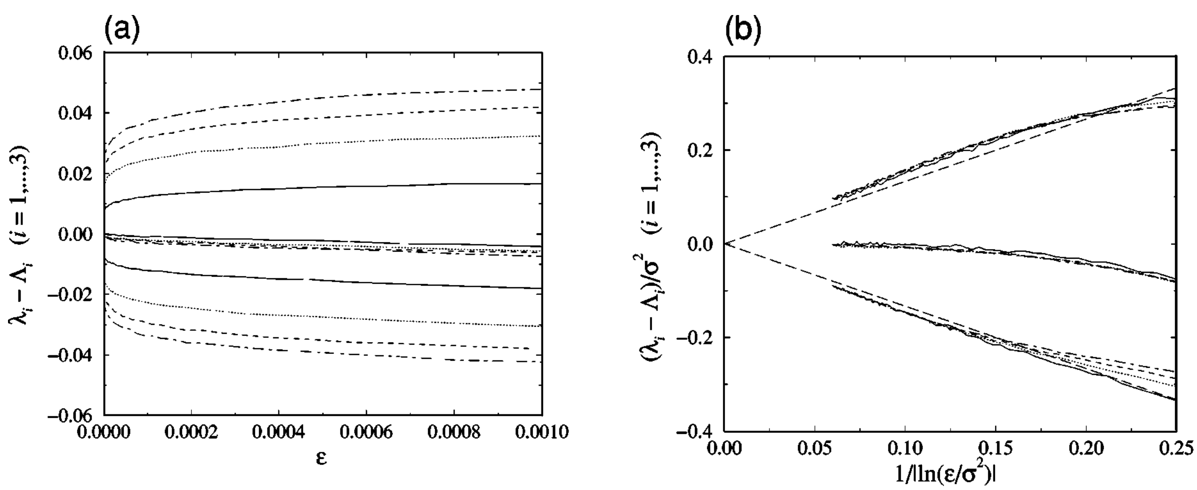

FIG. 8. Three coupled skewed Bernoulli maps. (a) The exponents $\lambda_i - \Lambda_i (i=1,\ldots,3)$ vs ε for $x_0=1/3$ (solid lines), $x_0=1/4$ (dotted lines), $x_0=1/5$ (dashed lines), and $x_0=1/6$ (dash-dotted lines). (b) The exponents $(\lambda_i - \Lambda_i)/\sigma^2 (i=1,\ldots,3)$ vs $1/|\ln(\varepsilon/\sigma^2)|$ for the same values of x_0 as in (a). The long-dashed lines show the analytical results (25).

if rather different systems are coupled, provided their Lyapunov exponents coincide. We also give a qualitative explanation of the effect, based on the interpretation of the perturbations' dynamics as coupled random walks. The coupling $\sim\varepsilon$ restricts the two-dimensional walk to a strip with a width $\sim\log\varepsilon$, with rather unusual ''reflection conditions'' on the strip borders. As a result the random walk (and, correspondingly, the Lyapunov exponent) gets a bias $\sim(\log\varepsilon)^{-1}$. It is not clear, if such an effect can be observed in the context of other random-walk-like phenomena.

We have also presented some generalizations where we do not have strict analytical results. For three coupled systems we were only able to obtain leading terms in the small coupling approximation; they are of the same inverse logarithm type as for two systems. Numerical simulations of a system with weaker stochastic properties (strange nonchaotic attractor) reveal, however, a power-law singularity, possibly due to the existence of long correlations in the dynamics of perturbations.

Recent results presented in Ref. [10] for coupled map lattices, including the case of fluctuating multiplier signs, support the assumption that the logarithmic singularity is a very general phenomenon of coupled chaotic systems.

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